

Reg. No.

--	--	--	--	--	--	--	--	--	--

**MTS 557****IV Semester M.Sc. Degree Examination, September/October 2022****MATHEMATICS****Algebraic Number Theory****(CBCS – New Syllabus)**

Time : 3 Hours

Max. Marks : 70

Note : 1) Answer **any five full** questions.2) Answer to **each full** question shall **not** exceed **eight** pages of the answer book. **No** additional sheets will be **provided** for answering.3) **Use** of scientific calculator is **permitted**.

1. a) State and prove Liouville's theorem for a real algebraic number of degree more than 1.
- b) Define the notions of trace and norm of an element in algebraic number field K . If K is an algebraic number field of degree n over $\mathbb{Q}$, then show that for any, $\alpha \in K$, $\text{Tr}_K(\alpha) = \sigma_1(\alpha) + \sigma_2(\alpha) + \dots + \sigma_n(\alpha)$, and $N_K(\alpha) = \sigma_1(\alpha) \sigma_2(\alpha) \dots \sigma_n(\alpha)$, where $\sigma_1, \sigma_2, \dots, \sigma_n$ are the n distinct $\mathbb{Q}$ -isomorphisms of K into $\mathbb{C}$. Deduce that for any algebraic integer α , $\text{Tr}_K(\alpha)$ and $N_K(\alpha)$ are integers.
- c) If $\xi = e^{\frac{2\pi i}{7}}$ and $K = \mathbb{Q}(\xi)$, then find $T_{K/\mathbb{Q}}(\xi + \xi^2 + \xi^3)$. **(5+7+2)**
2. a) Find the units in the ring of algebraic integers $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{d})$ and d is a negative square free integer.
- b) Prove that every algebraic number field has an integral basis.
- c) Determine the discriminant of $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. **(6+5+3)**
3. a) Prove that factorization into irreducibles is not unique in the ring of integers of $\mathbb{Q}(\sqrt{-5})$.
- b) Show that $K = \mathbb{Q}(\sqrt{d})$ is norm-Euclidean for $d = -1, -2, -3, -7$ and -11 .
- c) Show that $x^2 + 5 = y^3$ has no integer solution. **(5+5+4)**

P.T.O.



4. Let K be an algebraic number field and let $\mathcal{O}_K$ be its ring of integers.
- If norm of an element x in $\mathcal{O}_K$ is a rational prime, prove that x is irreducible in $\mathcal{O}_K$.
 - Prove that $\mathcal{O}_K$ is a Dedekind domain. (2+12)
5. a) Let $\mathcal{O}_K$ be the ring of integers of an algebraic number field K . Let P be a non-zero prime ideal of $\mathcal{O}_K$ and let P^{-1} denotes the set $\{\alpha \in K : \alpha P \subseteq \mathcal{O}_K\}$. Show that P^{-1} is a fractional ideal of $\mathcal{O}_K$, $\mathcal{O}_K \subseteq P^{-1}$ and $PP^{-1} = \mathcal{O}_K$.
- b) Find the inverse of ideal $\langle 2, \sqrt{6} \rangle$ in $D = \mathbb{Z} + \mathbb{Z}\sqrt{6}$. (10+4)
6. a) If I and J are ideals of a Dedekind domain R , prove that I divides J if and only if $J \subseteq I$. Hence deduce that if P is prime ideal and $P|IJ$, then $P|I$ or $P|J$.
- b) Prove or disprove the following :
- Every unique factorization domain is integrally closed.
 - Every principal ideal domain is a Dedekind domain.
 - Every Dedekind domain is a principal ideal domain.
- c) Find the prime factorization of the ideal $68\mathbb{Z}$ in the ring $\mathbb{Z}$. (4+8+2)
7. a) Let K be an algebraic number field and let $\mathcal{O}_K$ be its ring of integers. Prove that any non-zero proper ideal of $\mathcal{O}_K$ has finitely many divisors.
- b) Let K be an algebraic number field of degree n . If a rational prime p ramifies in K , then prove that p divides d_K , the discriminant of K .
- c) Let $K = \mathbb{Q}(\theta)$, where θ is a root of the polynomial $x^3 - 18x - 6$. Find out how the rational primes 2, 3, 5 and 11 split in K . (5+5+4)
8. a) Define the class-group and the class-number of an algebraic number field K . Prove that $\mathcal{O}_K$, the ring of integers of K is unique factorization domain if and only if the class-number is 1.
- b) Determine the class number of $K = \mathbb{Q}(\sqrt{-19})$. (8+6)
-