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**MTH 453****II Semester M.Sc. Degree Examination, September/October 2022****MATHEMATICS****Real Analysis – II****(CBCS – New Syllabus)**

Time : 3 Hours

Max. Marks : 70

Note : 1) Answer **any five full** questions.2) **No** additional sheets will be provided for answering.3) Use of scientific calculator is **permitted**.

1. a) Let f be a bounded real function on $[a, b]$ and α be a monotonically increasing real function on $[a, b]$. If f is monotonic on $[a, b]$, then prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$.
- b) Let f be a bounded real function on $[a, b]$ and α be a monotonically increasing real function on $[a, b]$. If f has only finitely many discontinuities on $[a, b]$ and α is continuous at every point of discontinuity of f on $[a, b]$, then prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. **(5+9)**
2. a) Suppose $\sum c_n$ converges, where $c_n \geq 0$ for $n \in \mathbb{N}$, $\{s_n\}$ is a sequence of distinct points in (a, b) and $\alpha(x) = \sum c_n I(x - s_n)$, where I denotes the unit step function. Let f be continuous on $[a, b]$. Then prove that $\int_a^b f d\alpha = \sum c_n f(s_n)$.
- b) If γ' is continuous on $[a, b]$, then show that γ is rectifiable and $A(\gamma) = \int_a^b |\gamma'(t)| dt$.
- c) Let f be a bounded real function on $[a, b]$ and suppose $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$ on $[a, b]$? Justify. **(6+6+2)**
3. a) Let f be a bounded real function on $[a, b]$ and α be a monotonically increasing real function on $[a, b]$. Suppose $\alpha' \in \mathcal{R}$ on $[a, b]$. Prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if $f\alpha' \in \mathcal{R}$ on $[a, b]$ and in that case show that $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.
- b) Suppose $f \geq 0$ is a continuous function on $[a, b]$ such that $\int_a^b f(x) dx = 0$. Prove that $f = 0$ on $[a, b]$. **(9+5)**

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4. a) Suppose $f \geq 0$ and monotonically decreasing on $[1, \infty)$. Then prove that $\int_1^\infty f \, dt$ converges if and only if $\sum f(n)$ converges.
- b) Show that $f(x) = \frac{\sin x}{x}$, $x \in [1, \infty)$ is integrable on $[1, \infty)$, but not absolutely. **(8+6)**
5. a) Suppose $f_n \rightarrow f$ uniformly on a set E in a metric space. Let x_0 be a limit point of E and suppose that $\lim_{t \rightarrow x_0} f_n(t) = A_n$ for all n . Then prove that $\{A_n\}$ converges and $\lim_{t \rightarrow x_0} f(t) = \lim_{n \rightarrow \infty} A_n$.
- b) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \dots$, and suppose $f_n \rightarrow f$ uniformly on $[a, b]$. Then show that $f \in \mathcal{R}(\alpha)$ on $[a, b]$, and $\int_a^b f \, d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n \, d\alpha$. **(8+6)**
6. a) Prove that there exists a real continuous function on the real line which is nowhere differentiable.
- b) If K is a compact metric space, if $f_n \in \mathcal{C}(K)$ for all $n \geq 1$, and if $\{f_n\}$ converges uniformly on K , then show that the family $\tau = \{f_n\}$ is equicontinuous on K . **(11+3)**
7. a) If f is a complex valued continuous function on $[a, b]$, then prove that there exists a sequence $\{P_n\}$ of polynomials with complex coefficients such that $\lim_{n \rightarrow \infty} P_n(x) = f(x)$ uniformly on $[a, b]$.
- b) If f is continuous on $[0, 1]$ and if $\int_0^1 f(x) x^n \, dx = 0$ for all $n = 1, 2, 3, \dots$, then prove that $f(x) = 0 \, \forall x \in [0, 1]$. **(12+2)**
8. a) Suppose f maps an open subset $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$. Then $f \in \mathcal{C}''(E)$ if and only if the partial derivatives $D_{j_i} f$ exist and are continuous on E for $1 \leq i \leq m, 1 \leq j \leq n$.
- b) If X is a complete metric space, and if ϕ is a contraction of X into X , then prove that there exists one and only one $x \in X$ such that $\phi(x) = x$. **(10+4)**
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