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MTH 452

II Semester M.Sc. Degree Examination, Sept./Oct. 2022

MATHEMATICS

Algebra – II

(CBCS – New Syllabus)

Time : 3 Hours

Max. Marks : 70

- Notes :** 1) Answer **any five full** questions.
2) **No** additional sheets will be provided for answering.
3) **Use** of scientific calculator is permitted.

1. a) Let D be an integral domain. For $a, b \in D$, prove or disprove that :
 - i) a divides b in D if and only if $\langle a \rangle \subseteq \langle b \rangle$.
 - ii) a is irreducible if and only if $\langle a \rangle$ is a maximal ideal in D .
 - iii) $\langle a \rangle$ and $\langle b \rangle$ are equal if and only if a and b are associates in D .
- b) Define a Euclidean domain and a principal ideal domain. Prove that every Euclidean domain is a principal ideal domain.
- c) Let a, b be elements of a principal ideal domain R , not both zero. Prove that greatest common divisor of a and b exists in R and it is unique up to associates. **(6+4+4)**
2. a) Let D be an integral domain. Then prove that the factoring into irreducible terminates in D if and only if D does not contain infinite strictly increasing chain $\langle a_1 \rangle \subsetneq \langle a_2 \rangle \subsetneq \langle a_3 \rangle \subsetneq \dots$ of principal ideals.
- b) Prove that, the product of any two primitive polynomials is primitive.
- c) Find all irreducible polynomials of degree less than 4 in $\mathbb{Z}_2[x]$. **(5+5+4)**
3. a) Let f be an integer polynomial with positive leading coefficient. Then prove that f is irreducible in $\mathbb{Z}[x]$ if and only if it is either a prime integer or a primitive polynomial that is irreducible in $\mathbb{Q}[x]$.
- b) Let α be Gauss prime and $\bar{\alpha}$ be its complex conjugate. Then prove that $\alpha\bar{\alpha}$ is either a prime integer or square of a prime integer.
- c) Factor $6 + 9i$ into primes in $\mathbb{Z}[i]$. **(5+6+3)**

P.T.O.



4. a) Prove or disprove the following :
- i) Every finite extension is an algebraic extension.
 - ii) Every algebraic extension is a finite extension.
- b) If L is a finite extension of K and K is a finite extension of F , then prove that L is a finite extension of F .
- c) Let $K|F$ be a field extension and $\alpha \in K$. Then prove that α is algebraic over F , if and only if $F[\alpha] = F(\alpha)$. (6+4+4)
5. a) Show that trisecting an angle is impossible using Ruler and Compass alone.
- b) If p is a prime number and a regular p -gon can be constructed by using ruler and compass, then show that $p = 2^k + 1$, for some integer $k \geq 0$.
- c) Find the splitting field and the degree of extension of the splitting field of $f(x) = x^4 + x^2 + 1$ over $\mathbb{Q}$. (6+4+4)
6. a) Prove that characteristic of any finite field is a prime p and it has p^n elements, where n is a positive integer.
- b) Let F be a field of order p^n , where p is prime and n is a positive integer. Prove that F contains a subfield of order p^k if and only if $k | n$.
- c) If F is a field of characteristic zero and $f(x)$ is an irreducible polynomial in $F[x]$, then prove that f has no multiple root in any extension of F . (5+5+4)
7. a) Define an algebraically closed field. Prove that a field F is algebraically closed if and only if every polynomial in $F[x]$ splits into linear factors in $F[x]$.
- b) Let F be a field of characteristic zero. Then prove that any finite extension of F is a simple extension.
- c) Show that $\mathbb{Q}(i, \sqrt[3]{2}) = \mathbb{Q}(i\sqrt[3]{2})$. Find $[\mathbb{Q}(i, \sqrt[3]{2}) : \mathbb{Q}]$. (3+7+4)
8. a) Define the fixed field of group of automorphisms of a field K . If K is a Galois extension of field F , then prove that the fixed field of Galois group $G(K/F)$ is F .
- b) State and prove the main theorem of Galois theory. (3+11)
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